Lecture 23 (11/17/21).

- Finish FEP homotopy, simple connectness + 2 cor's from Lecture 22 notes.

Counting zeros.

Argument Principle. Let f be analytic in G, $\alpha \in \mathbb{C}$ and $\{a_1, a_2, \dots\}$ the seq. of points in G s.t. $f(z)=\alpha$ (seq. could be finite or infinite and roots appear w/ multiplicity). If γ is a closed curve in G s.t. $n(\gamma, z)=0$, $\forall z \in \mathbb{C} \setminus G$ ($\gamma \approx 0$) and if no $a_j \in \gamma$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz = \sum_{j=1}^{\infty} n(\gamma, a_j).$$

Pf. WLOG. $\alpha=0$. We may also assume that the seq. of roots is finite since only a finite number of the a_j can have $n(\gamma, a_j) \neq 0$. (Why? DIY).

By previous, we can then factor f:
$f(z) = (z-a_1)\dots(z-a_n)\,g(z)$, where g is analytic in G and $g \neq 0$. We compute

$$f'(z) = \sum_{j=1}^{n} (z-a_1)\dots\widehat{(z-a_j)}\dots(z-a_n)\,g(z) +$$

factor omitted

$$+ (z-a_1)\dots(z-a_n)\,g'(z) \Rightarrow$$

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z-a_j} + \frac{g'(z)}{g(z)}$$

$$\Rightarrow$$

$$\frac{1}{2\pi i}\int \frac{f'}{f}\,dz = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a_j} + \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{g'}{g}\,dz}_{\substack{=0 \text{ by CT since} \\ g\neq 0 \text{ in } G.}}$$

$$= \sum_{j=1}^{n} n(\gamma, a_j) \quad \text{as claimed.}$$

∎

<u>Rem</u>. Why is this called Argument Principle? Same reason the index of γ wrt a point a is called winding #.
Let $\sigma = f \circ \gamma$ in Arg Princ. Thm, then

$$n(\sigma, \alpha) = \frac{1}{2\pi i} \int_{\sigma} \frac{dw}{w-\alpha} = \left\{ \begin{array}{l} w = f(z) \\ dw = f'(z)\,dz \end{array} \right\}$$

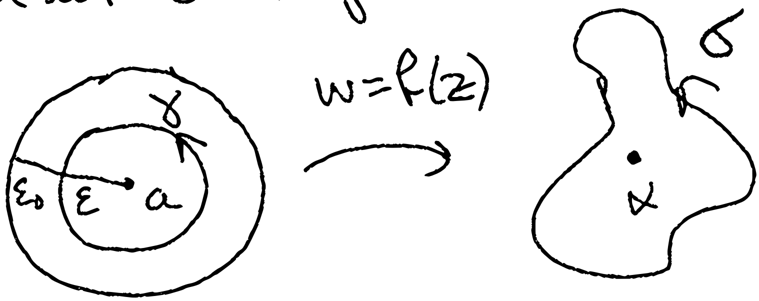
$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)\,dz}{f(z)-\alpha} \underset{\text{by AP}}{=} \sum_{j=1}^{\infty} n(\gamma, a_j).$$

In other words, when γ is a simple closed curve so that

$n(\gamma, a_j) = 1$ if a_j "inside" γ and $n(\gamma, a_j) = 0$ if a_j is "outside", then # of zeros (w/ multipl.) of $f - \alpha$ equals the winding # of $\sigma = f \circ \gamma$ w.r.t. α.

Thm 1. Let f be analytic in G, $a \in G$ and $\alpha = f(a)$ and m = multi of zero of $f - \alpha$ at $z = a$. Then $\exists\, \varepsilon_0 > 0$ s.t. $\forall\, 0 < \varepsilon < \varepsilon_0$ $\exists\, \delta > 0$ s.t. $f(z) = \beta$ has precisely m simple roots in $B(a, \varepsilon)$ for every $\beta \in B(\alpha, \delta) \setminus \{\alpha\}$. (simple root $\Leftrightarrow$ multi of zero of $f - \beta$ is one.)

Pf. Choose $\varepsilon_0 > 0$ so small that $f - \alpha$ has only the zero $z = a$ in $B(a, \varepsilon_0)$ and $f' \neq 0$ in $B(a, \varepsilon_0) \setminus \{a\}$. (Why is this possible? DIY.) Let γ be circle ↺ of radius $\varepsilon < \varepsilon_0$ centered at $z = a$ and let $\sigma = f \circ \gamma$.



Since $\alpha \notin \sigma$ (by choice of ε_0) and σ compact $\exists\, B(\alpha, \delta)$ s.t. $\overline{B(\alpha, \delta)} \cap \sigma = \emptyset$.

In particular, $B(\alpha,\delta)$ is contained in one component of $\mathbb{C}\sim\sigma$, so by previous $n(\sigma,\beta)=n(\sigma,\alpha)$, $\forall\beta\in B(\alpha,\delta)$. But, by AP, $n(\sigma,\alpha)=m$ (again by choice of ε_0) and thus, again by AP and the calculation in the remark following it, $f-\beta$ has m roots in $B(a,\varepsilon)$, counting multi. But, by previous, the multi of any root must be <u>one</u> since $f'\neq 0$ in $B(a,\varepsilon)$.

Two corollaries follow from Thm 1 + previous.

<u>Open Mapping Thm</u>. Let f be analytic and nonconstant in a region G. Then, $f(G)$ is open.

<u>Cor 1</u>. If f is analytic in region G and 1:1, then $\Omega=f(G)$ is open and $f^{-1}: \Omega\to G$ is analytic.